On-Line Geometric Modeling Notes

BÉZIER CURVES

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Overview

The Bézier curve representation is one that is utilized most frequently in computer graphics and geometric modeling. The curve is defined geometrically, which means that the parameters have geometric meaning – they are just points in three-dimensional space. It was developed by two competing European engineers in the late 1960s to attempt to draw automotive components.

In these notes, we develop the mathematical description for the Bézier curve of arbitrary degree by generalizing the development for the quadratic and cubic Bézier curves, creating a parameterized version of the curve.

Specification of the Curve

Given the set of control points, $\{\mathbf{P}_0, \mathbf{P}_1, ..., \mathbf{P}_n\}$, we can define a Bézier curve of degree n by either of the following definitions:

The Analytic Definition

$$\mathbf{P}(t) \;=\; \sum_{i=0}^{n} \mathbf{P}_i B_{i,n}(t)$$

where

$$B_{i,n}(t) \;=\; \binom{n}{i} t^i (1-t)^{n-i}$$

are the Bernstein polynomials of degree n, and t ranges between zero and one – $0 \leq t \leq 1$.

Geometric Definition

$$\mathbf{P}(t) \;=\; \mathbf{P}_n^{(n)}(t)$$

where

$$\mathbf{P}_i^{(j)}(t) \;=\; \begin{cases} (1-t)\mathbf{P}_{i-1}^{(j-1)}(t) \;+\; t\mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0, \\ \mathbf{P}_i & \text{otherwise} \end{cases}$$

where t ranges between zero and one – $0 \le t \le 1$.

Properties of the Bézier Curve

The Bézier curve has properties similar to that of the quadratic and cubic curve. These can be verified directly from the equations above.

- $\mathbf{P}_0$ and $\mathbf{P}_n$ are on the curve.
- The curve is continuous and has continuous derivatives of all orders.
- The tangent line to the curve at the point $\mathbf{P}_0$ is the line $\overline{\mathbf{P}_0\mathbf{P}_1}$. The tangent to the curve at the point $\mathbf{P}_n$ is the line $\overline{\mathbf{P}_{n-1}\mathbf{P}_n}$.
- The curve lies within the convex hull of its control points. This is because each successive $\mathbf{P}_i^{(j)}$ is a convex combination of the points $\mathbf{P}_i^{(j-1)}$ and $\mathbf{P}_{i-1}^{(j-1)}$.
- $\mathbf{P}_1$, $\mathbf{P}_2$, ..., $\mathbf{P}_{n-1}$ are all on the curve only if the curve is linear.

Summary

Given a sequence of $n+1$ control points, one can specify a Bézier curve of degree n defined by these points. Two definitions of the curve can be given: an analytic definition specifying the blending of the control points with Bernstein polynomials, and a geometric definition specifying a recursive generation procedure that calculates successive points on line segments developed from the control point sequence.