

On-Line Geometric Modeling Notes

DEFINITION OF A B-SPLINE CURVE

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Overview

These notes present the direct definition of the B-Spline curve. This definition is given in two ways: first by an analytical definition using the normalized B-spline blending functions, and then through a geometric definition.

The B-Spline Curve – Analytical Definition

A B-spline curve $\mathbf{P}(t)$, is defined by

$$\mathbf{P}(t) = \sum_{i=0}^n \mathbf{P}_i N_{i,k}(t)$$

where

- the $\{\mathbf{P}_i : i = 0, 1, \dots, n\}$ are the control points,
- k is the order of the polynomial segments of the B-spline curve. Order k means that the curve is made up of piecewise polynomial segments of degree $k - 1$,
- the $N_{i,k}(t)$ are the “normalized B-spline blending functions”. They are described by the order k and by a non-decreasing sequence of real numbers

$$\{t_i : i = 0, \dots, n + k\}.$$

normally called the “knot sequence”. The $N_{i,k}$ functions are described as follows

$$N_{i,1}(t) = \begin{cases} 1 & \text{if } u \in [t_i, t_{i+1}), \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

and if $k > 1$,

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) \quad (2)$$

- and $t \in [t_{k-1}, t_{n+1})$.

We note that if, in equation (2), either of the N terms on the right hand side of the equation are zero, or the subscripts are out of the range of the summation limits, then the associated fraction is not evaluated and the term becomes zero. This is to avoid a zero-over-zero evaluation problem. We also direct the readers attention to the “closed-open” interval in the equation (1).

The order k is independent of the number of control points ($n + 1$). In the B-Spline curve, unlike the Bézier Curve, we have the flexibility of using many control points, and restricting the degree of the polynomial segments.

The B-Spline Curve – Geometric Definition

Given a set of Control Points $\{\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_n\}$, an order k , and a set of knots $\{t_0, t_1, \dots, t_{n+k}\}$, the B-Spline curve of order k is defined to be

$$\mathbf{P}(t) = \mathbf{P}_l^{(k-1)}(t) \text{ if } u \in [t_l, t_{l+1})$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1 - \tau_i^j) \mathbf{P}_{i-1}^{(j-1)}(t) + \tau_i^j \mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0, \\ \mathbf{P}_i & \text{if } j = 0. \end{cases}$$

and

$$\tau_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$$

