

## On-Line Geometric Modeling Notes

# CUBIC BÉZIER CURVES

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### Overview

The Bézier curve representation is one that is utilized most frequently in computer graphics and geometric modeling. The curve is defined geometrically, which means that the parameters have geometric meaning – they are just points in three-dimensional space. It was developed by two competing European engineers in the late 1960s to attempt to draw automotive components.

In these notes, we develop the cubic Bézier curve. This curve can be developed through a divide-and-conquer approach similar to the quadratic curve. However, in these notes, we will develop a parameterized version of the curve which proceeds almost identically to the development for the quadratic Bézier curve

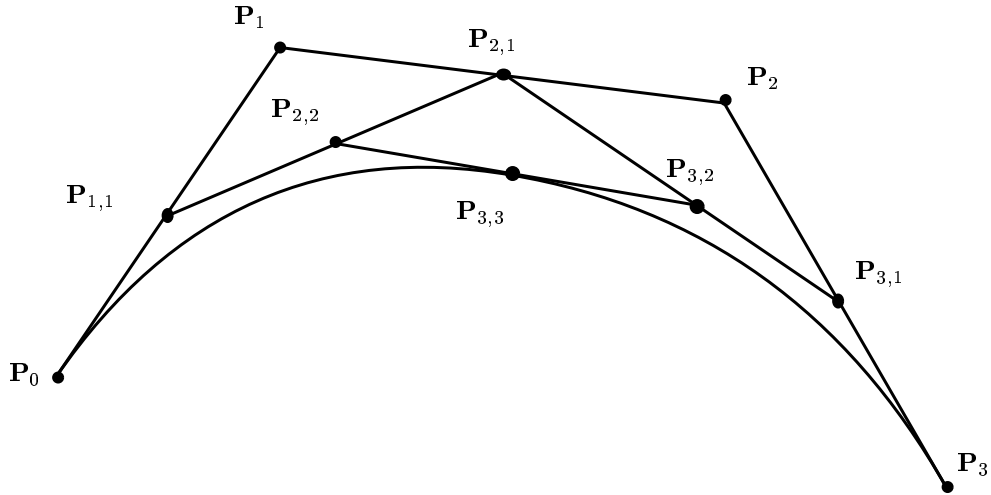
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### Defining The Cubic Bézier Curve

Given four control points,  $\mathbf{P}_0$ ,  $\mathbf{P}_1$ ,  $\mathbf{P}_2$ ,  $\mathbf{P}_3$ , one can generate a curve  $\mathbf{P}(t)$ , as we did in the case of the quadratic Bézier curve, by

- let  $\mathbf{P}_1^{(1)}(t) = t\mathbf{P}_1 + (1-t)\mathbf{P}_0$
- let  $\mathbf{P}_2^{(1)}(t) = t\mathbf{P}_2 + (1-t)\mathbf{P}_1$
- let  $\mathbf{P}_3^{(1)}(t) = t\mathbf{P}_3 + (1-t)\mathbf{P}_2$
- let  $\mathbf{P}_2^{(2)}(t) = t\mathbf{P}_2^{(1)}(t) + (1-t)\mathbf{P}_1^{(1)}(t)$
- let  $\mathbf{P}_3^{(2)}(t) = t\mathbf{P}_3^{(1)}(t) + (1-t)\mathbf{P}_2^{(1)}(t)$
- let  $\mathbf{P}_3^{(3)}(t) = t\mathbf{P}_3^{(2)}(t) + (1-t)\mathbf{P}_2^{(2)}(t)$
- $\mathbf{P}_3^{(3)}(t)$  is defined to be  $\mathbf{P}(t)$

This construction is shown in the figure below



notice that we did the same process as in the quadratic Bézier curve, but did one additional level. The procedure, as in the quadratic case, produces a point on the curve and subdivides the curve by producing 2 new sets of 4 control points.

Simplifying the above construction, we have

$$\begin{aligned}
 \mathbf{P}(t) &= \mathbf{P}_3^{(3)}(t) \\
 &= t\mathbf{P}_3^{(2)}(t) + (1-t)\mathbf{P}_2^{(2)}(t) \\
 &= t \left[ t\mathbf{P}_3^{(1)}(t) + (1-t)\mathbf{P}_2^{(1)}(t) \right] \\
 &\quad + (1-t) \left[ t\mathbf{P}_2^{(1)}(t) + (1-t)\mathbf{P}_1^{(1)}(t) \right] \\
 &= t^2\mathbf{P}_3^{(1)}(t) + 2t(1-t)\mathbf{P}_2^{(1)}(t) + (1-t)^2\mathbf{P}_1^{(1)}(t) \\
 &= t^2 [t\mathbf{P}_3 + (1-t)\mathbf{P}_2] + 2t(1-t) [t\mathbf{P}_2 + (1-t)\mathbf{P}_1] \\
 &\quad + (1-t)^2 [t\mathbf{P}_1 + (1-t)\mathbf{P}_0] \\
 &= t^3\mathbf{P}_3 + 3t^2(1-t)\mathbf{P}_2 + 3t(1-t)^2\mathbf{P}_1 + (1-t)^3\mathbf{P}_0
 \end{aligned}$$

which is the analytic form of the curve.

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### Summarizing the Development of the Curve

As in the quadratic case, we have developed two methods for generating points on the curve.

- **The Geometric Method** – Given the control points  $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ , and a value  $t \in [0, 1]$ , we can generate the point  $\mathbf{P}(t)$  on the Bézier curve by

$$\mathbf{P}(t) = \mathbf{P}_3^{(3)}(t)$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-t)\mathbf{P}_{i-1}^{(j-1)}(t) + t\mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0 \\ \mathbf{P}_i & \text{otherwise} \end{cases}$$

- **The Analytic Method** – Given the control points  $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ , we define the Bézier curve to be

$$\mathbf{P}(t) = \sum_{i=0}^3 \mathbf{P}_i B_{i,3}(t)$$

where

$$B_{0,3}(t) = (1-t)^3$$

$$B_{1,3}(t) = 3t(1-t)^2$$

$$B_{2,3}(t) = 3t^2(1-t)$$

$$B_{3,3}(t) = t^3$$

the Bernstein polynomials of degree three.

### Properties of the Cubic Bézier Curve

The cubic Bézier curve has properties similar to that of the quadratic curve. These can be verified directly from the equations above.

- $\mathbf{P}_0$  and  $\mathbf{P}_3$  are on the curve.

- The curve is continuous, infinitely differentiable, and the second derivatives are continuous (automatic for a polynomial curve).
  - The tangent line to the curve at the point  $\mathbf{P}_0$  is the line  $\overline{\mathbf{P}_0\mathbf{P}_1}$ . The tangent to the curve at the point  $\mathbf{P}_3$  is the line  $\overline{\mathbf{P}_2\mathbf{P}_3}$ .
  - The curve lies within the convex hull of its control points. This is because each successive  $\mathbf{P}_i^{(j)}$  is a convex combination of the points  $\mathbf{P}_i^{(j-1)}$  and  $\mathbf{P}_{i-1}^{(j-1)}$ .
  - Both  $\mathbf{P}_1$  and  $\mathbf{P}_2$  are on the curve only if the curve is linear.
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### Summary

The procedure for developing the cubic Bézier curve is nearly identical to that for the quadratic curve – the primary difference is that we have four control points and must proceed one additional level in the recursion to get a point on the curve. This procedure is extendable so that Bézier curves can be developed for any number of control points.

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