On-Line Geometric Modeling Notes

THE UNIFORM B-SPLINE BLENDING FUNCTION

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Overview

The uniform B-splines are based upon a knot sequence that has uniform spacing. This implies that the uniform B-spline blending functions $N_{i,k}(t)$ are all translates of a single blending function $N_k(t)$ where

$$N_{i,k}(t) = N_k(t-i)$$

This single blending function can be defined by convolution of blending functions of lower degree. This is the topic of these notes.

Definition of the Blending Functions Utilizing Convolution

The uniform kth order B-spline blending function N_k is defined recursively by

$$N_1(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$N_k(t) = (N_{k-1} * N_1)(t)$$

That is, the kth order blending function is defined by convolving the $k-1$st order blending function with the first order blending function. This convolution can be seen to be the integral

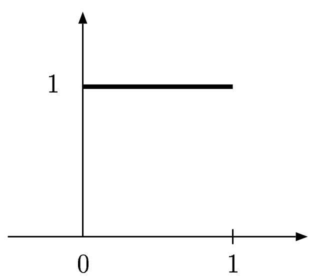
$$
\begin{aligned}
N_k(t) &= (N_{k-1} * N_1)(t) \\
&= \int_{-\infty}^{\infty} N_{k-1}(x) N_1(t-x)dx \\
&= \int_{t-1}^{t} N_{k-1}(x)dx
\end{aligned}
$$

The First Order Blending Function

The first order blending function is just the Haar scaling function

$$
N_1(t) \;=\; \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

and is shown by the graph



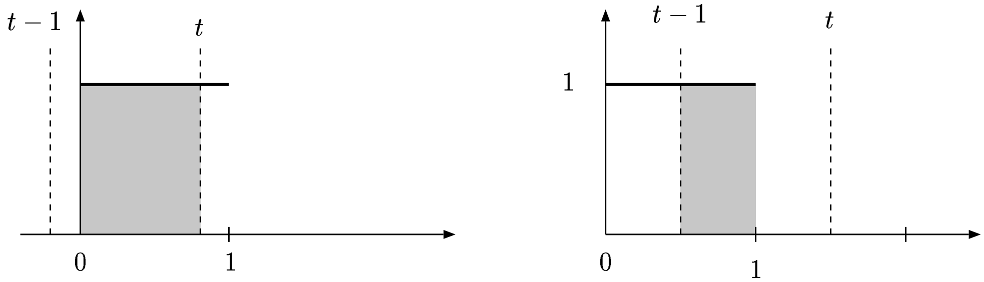
The support of this function is the interval $[0, 1]$.

The Second Order Blending Function

To calculate the second order blending function we must calculate

$$
N_2(t) \;=\; \int_{t-1}^{t} N_1(x)dx
$$

The function $N_1(x)$ is nonzero only when $0 \le x \le 1$. Thus, we can get nonzero values in the integrand $N_1(x)$ for any t where $0 < t < 2$. The integral splits naturally into the two cases shown below – for $0 \le t \le 1$ and $1 \le t \le 2$.

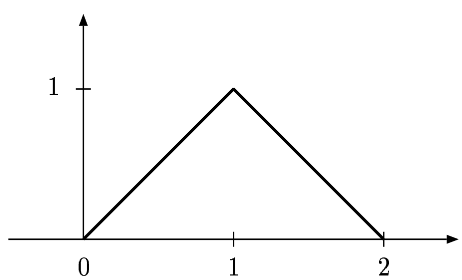


where in each case we have shaded the areas between the limits of integration 0 and 1.

So we have that

$$
\begin{aligned}
N_2(t) &= \int_{t-1}^{t} N_1(x)dx \\
&= \begin{cases} \displaystyle\int_0^t dx & \text{if } 0 \le t \le 1 \\ \displaystyle\int_{t-1}^{1} dx & \text{if } 1 \le t \le 2 \end{cases} \\
&= \begin{cases} t & \text{if } \ 0 \le t \le 1 \\ 2-t & \text{if } \ 1 \le t \le 2 \end{cases}
\end{aligned}
$$

which is illustrated by



It is clear that the support of $N_2(t)$ is the interval $[0, 2]$

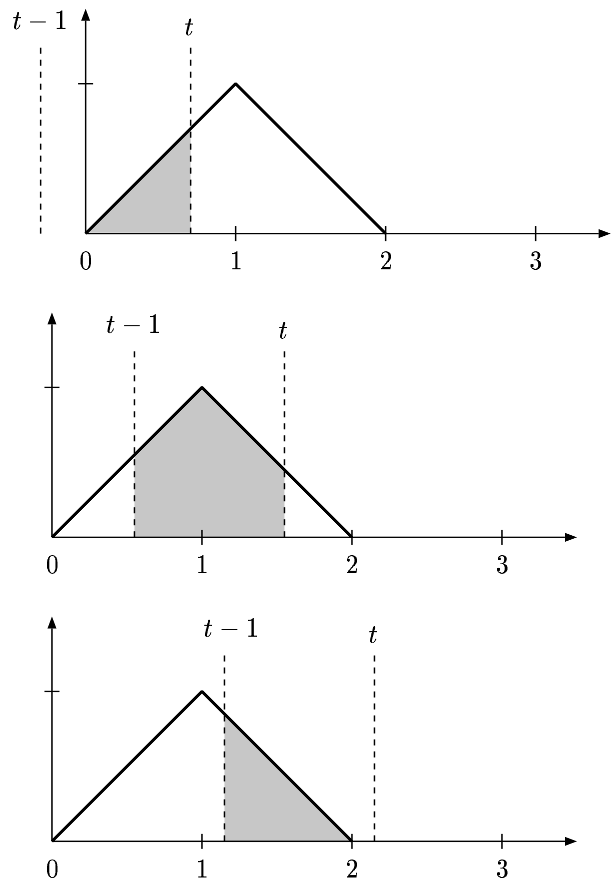
The Third Order Blending Function

To calculate the third order blending function, we must calculate

$$N_3(t) \ = \ \int_{t-1}^{t} N_2(x)dx$$

The function $N_2(x)$ is nonzero only when $0 \le x \le 2$, so we can get nonzero values in the integrand for any t where $0 < t < 3$.

This is straightforward to calculate once the reader sees that there are three cases, each depending on t. These three cases are illustrated below as

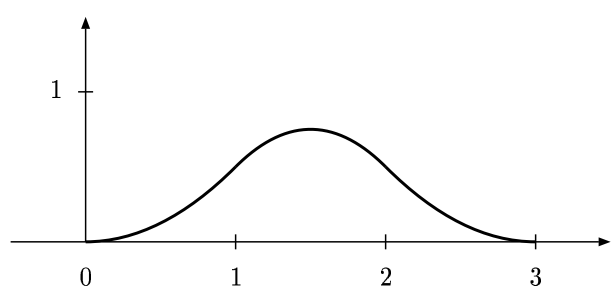


In each case the section of the curve $N_2(x)$ that lies between the integration bounds of 0 and 1 has been shaded.

So now we can calculate the integral by

$$
\begin{aligned}
N_3(t) &= \int_{t-1}^{t} N_2(x)dx \\
&= \begin{cases} \int_0^t N_2(x)dx & \text{if } 0 \le t \le 1 \\ \int_{t-1}^{1} N_2(x)dx \ + \int_1^t N_2(x)dx & \text{if } 1 \le t \le 2 \\ \int_{t-2}^{1} N_2(x)dx & \text{if } 2 \le t \le 3 \end{cases} \\
&= \begin{cases} \int_0^t xdx & \text{if } 0 \le t \le 1 \\ \int_{t-1}^{1} 2 - xdx \ + \int_1^t xdx & \text{if } 1 \le t \le 2 \\ \int_{t-2}^{1} 2 - xdx & \text{if } 2 \le t \le 3 \end{cases} \\
&= \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \le t \le 1 \\ \frac{1}{2}\left(-2t^2 + 6t - 3\right) & \text{if } 1 \le t \le 2 \\ \frac{1}{2}\left(t^2 - 6t + 9\right) & \text{if } 2 \le t \le 3 \end{cases}
\end{aligned}
$$

This curve is a *piecewise quadratic* – i.e. it has quadratic pieces that are smoothly joined together. The curve is drawn as



It is clear that the support of $N_3(t)$ is the interval $[0, 3]$

Summary

The uniform B-spline is somewhat unique as all blending functions are given as a translate of only one function. We have shown here that this single blending function can be calculated in an interesting way using convolution.

References

[1] BARTELS, R., BEATTY, J., AND BARSKY, B. *An Introduction to Splines for Use in Computer Graphics and Geometric Modeling*. Morgan Kaufmann Publishers, Palo Alto, CA, 1987.

[2] UEDA, M., AND LODHA, S. Wavelets: An elementary introduction and examples. Technical Report UCSC-CRL-94-47, Jan. 1994.
