On-Line Geometric Modeling Notes

THE TWO-SCALE RELATION FOR UNIFORM B-SPLINES

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Overview

The uniform B-splines are based upon a knot sequence that has uniform spacing. This implies that the uniform B-spline blending functions $N_{i,k}(t)$ are all translates of a single blending function $N_k(t)$ where

$$N_{i,k}(t) = N_k(t-i)$$

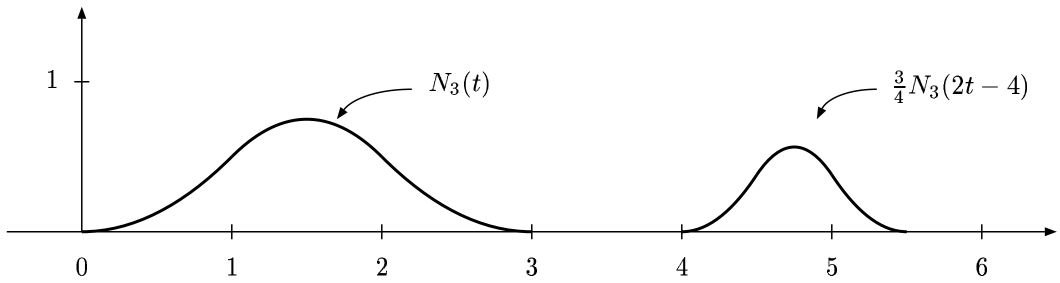
A remarkable property of this single blending function is that it can be written as a sum of scaled and translated copies of itself. Such a property is called a *two-scale relation* and is essential to defining wavelets on spaces of functions.

Translating and Scaling the Blending Function

The uniform B-spline blending function $N_k(t)$ can be scaled and translated simply by redefining the parameterization of the function. For example the function

$$\frac{3}{4}N_3(2t-4)$$

which is shown in the figure below (in relation to the blending function $N_3(t)$), translates the blending function so that its support begins at $t = 4$, and ends a $t = 5.5$, and the height of the function has been scaled by $.75$.



In general, the function $cN_k(at+b)$ has support over the interval $[b, b+\frac{k}{a}]$ and has the height of the function scaled by c.

The Two-Scale Relation for Uniform B-Splines

Given the general B-Spline blending function of order k, the two-scale relation is written as

$$N_k(t) \;=\; \sum_{i=0}^{m} p_i N_k(2t-i)$$

where

$$p_i \;=\; \frac{1}{2^{k-1}}\binom{k}{i}$$

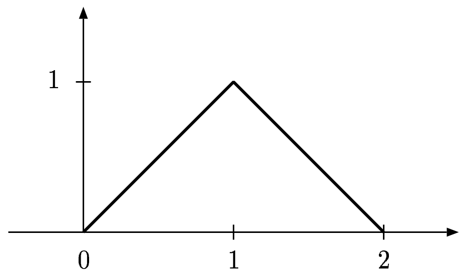
That is, we can take translated and scaled copies of the basic function, add them together, and get the basic function back. The development of the coefficients utilizes the fact that the uniform B-spline blending function can be defined by convolution.

The Two-Scale Relation for Uniform Linear B-Splines

The uniform 2nd order B-spline blending function $N_2(t)$ is defined by

$$N_2(t) \;=\; \begin{cases} t & \text{if } 0 \le t \le 1 \\ 2-t & \text{if } 1 \le t \le 2 \end{cases}$$

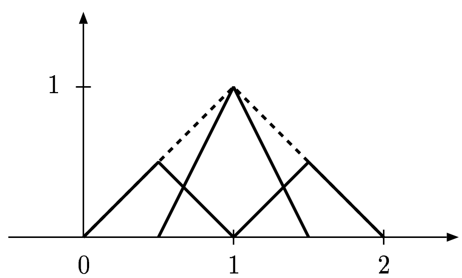
which is illustrated by



The two-scale relation for this function is given by

$$N_2(t) \;=\; \frac{1}{2}N_2(2t) + N_2(2t-1) + \frac{1}{2}N_2(2t-2)$$

The four components of this equation are shown in the following figure, where the original blending function is shown with dashed lines and the three scaled and translated functions are shown using solid lines.



The original blending function is obtained by summing the three scaled and translated functions at each point.

The Two-Scale Relation for Uniform Quadratic B-Splines

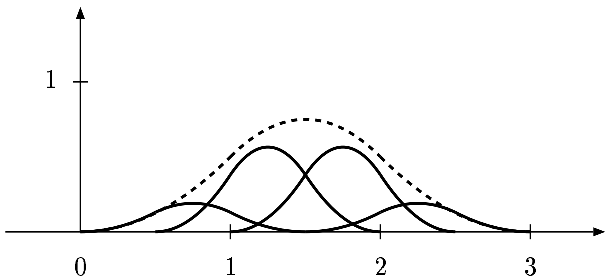
A less obvious example is given by the quadratic blending function. This 3rd order B-spline blending function $N_3(t)$ is defined by

$$N_3(t) \;=\; \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \le t \le 1 \\ \frac{1}{2}\left(-2t^2 + 6t - 3\right) & \text{if } 1 \le t \le 2 \\ \frac{1}{2}\left(t^2 - 6t + 9\right) & \text{if } 2 \le t \le 3 \end{cases}$$

The two-scale relation for this function is given by

$$N_3(t) \ = \ \frac{1}{4}N_3(2t) + \frac{3}{4}N_3(2t-1) + \frac{3}{4}N_3(2t-2) + \frac{1}{4}N_3(2t-3)$$

The five components of this equation are shown in the following figure, where the original blending function is shown with dashed lines and the four scaled and translated functions are shown using solid lines.



The original blending function is obtained by summing the four scaled and translated functions at each point.

Summary

The two-scale relation is an important identity when dealing with uniform B-splines (especially in relation to the definitions of B-spline wavelets), and is not easily duplicated with non-uniform splines. The proof of the general identity is also interesting as it uses the fact that the blending function can be defined using convolution.

References

[1] BARTELS, R., BEATTY, J., AND BARSKY, B. *An Introduction to Splines for Use in Computer Graphics and Geometric Modeling*. Morgan Kaufmann Publishers, Palo Alto, CA, 1987.

[2] UEDA, M., AND LODHA, S. Wavelets: An elementary introduction and examples. Technical Report UCSC-CRL-94-47, Jan. 1994.

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