GENERAL ROTATION ABOUT AN AXIS

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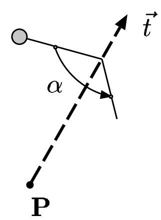
Overview

An axis in space is specified by a point $\mathbf{P}$ and a vector direction $\vec{t}$. Suppose that we wish to rotate an object about this arbitrary axis.



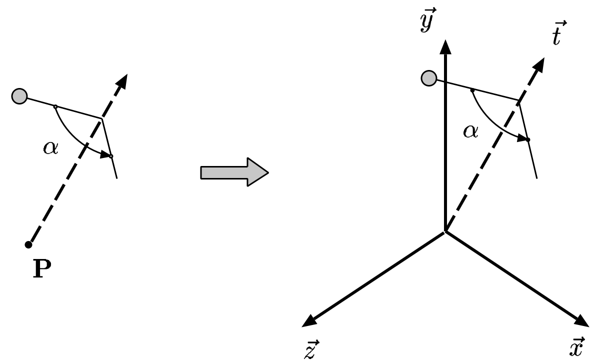
We know how to do this in the cases that the axis is the x axis, the y axis, or the z axis in the Cartesian frame (these were just generalizations of the two-dimensional rotations), but the general case is more difficult. In these notes we present a solution to this problem that utilizes both translation and the above rotation matrices to accomplish this task. (One can also approach this problem through the use of frame-to-frame-conversion transformations.)

Developing the General Rotation Matrix

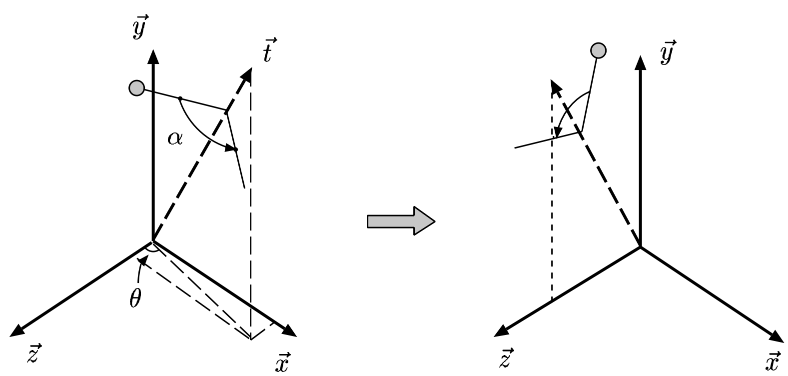
First assume that the axis of rotation can be specified in terms of Cartesian coordinates, i.e. can be represented by the point $\mathbf{P} = (x_p, y_p, z_p)$ and the vector $\vec{t} =< x_t, y_t, z_t >$. Then a rotation of α degrees about this axis can be defined by concatenating the following transformations

- Translate so that the point $\mathbf{P}$ moves to the origin

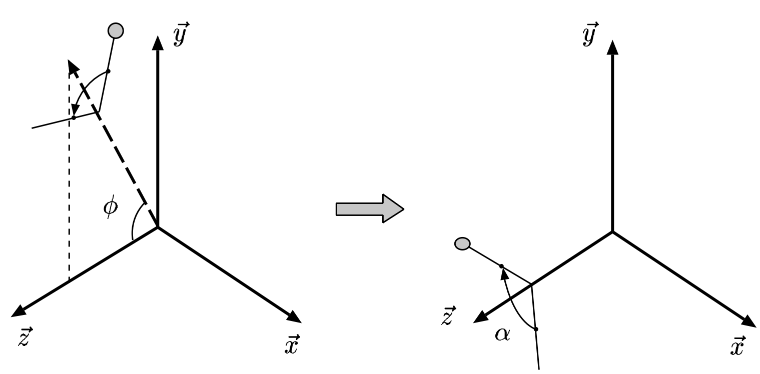
$$T_{(-x_p,-y_p,-z_p)},$$



- Use the elementary rotation transformations to rotate the vector until it coincides with one of the coordinate axes. To do this, first rotate the vector until it is in the yz plane by using a rotation $R_{y;-\theta}$ about the y axis.

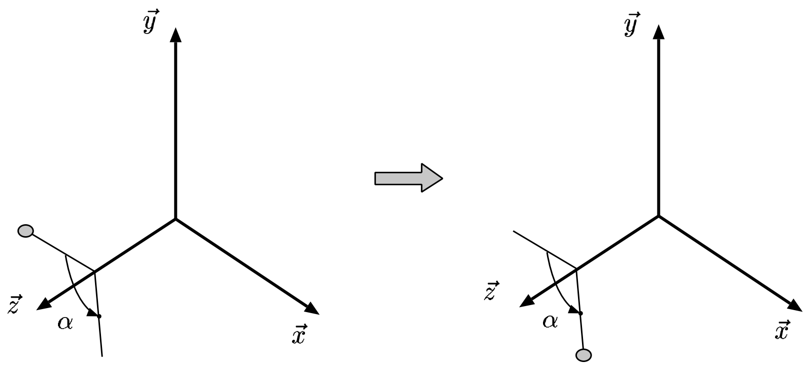


where $\theta = \arctan\left(\frac{x_v}{z_v}\right)$, and then use an x-axis rotation, of $R_{x;\phi}$, to rotate the vector until it coincides with the z axis.

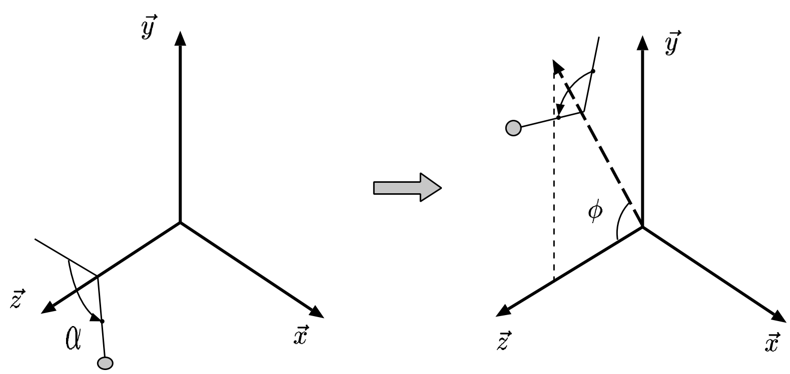


where $\phi = \arctan(\frac{y_v}{\sqrt{x_v^2+z_v^2}})$

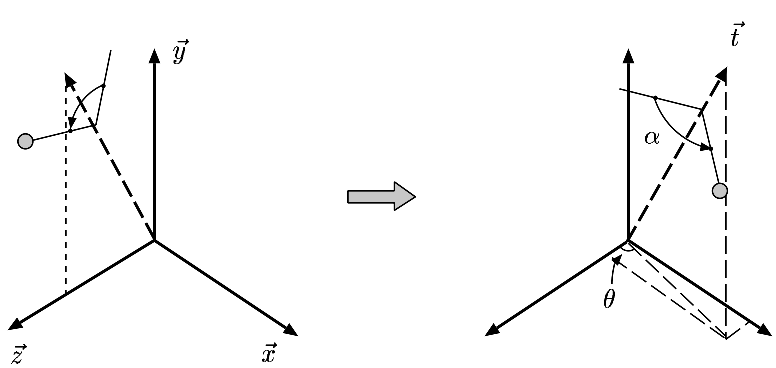
- Then use an α rotation about the z axis, $R_{z;\alpha}$.



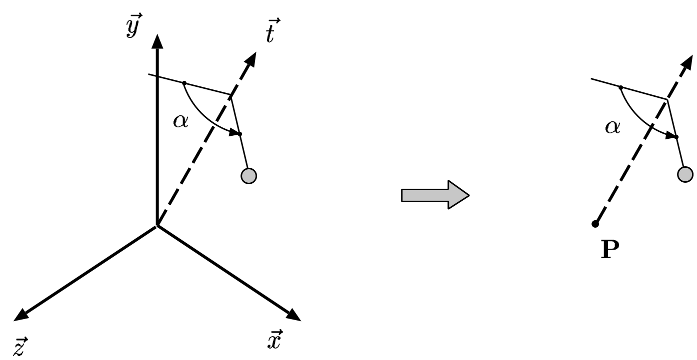
- Use rotations and translations to reverse the first two processes: First by a rotation $R_{x;-\phi}$, about the x axis



then by a rotation $R_{y;\theta}$ about the y axis



and finally using the translation, $T_{(x_p,y_p,z_p)}$ to translate back to the original axis.



The matrix representation of the general rotation is given by the product of the above transformations.

$$T_{(-x_p,-y_p,-z_p)}R_{y;-\theta}R_{x;\phi}R_{z;\alpha}R_{x;-\phi}R_{y;\theta}T_{(x_p,y_p,z_p)}$$

These can be multiplied together (they are all 4×4 matrices) to give one 4×4 matrix which represents the general rotation.

What if the Axis was Specified in a Local Frame?

In this case, we just convert the coordinates of the point $\mathbf{P}$ and vector $\vec{t}$ defining the axis to Cartesian coordinates using the frame-to-Cartesian-frame transformation, do the above operations, and then use the Cartesian-frame-to-frame to convert the resulting coordinates back to the local system.

Summary

We have developed a simple method using only basic transformations by which general rotation can be accomplished. It utilizes translation and the basic rotations about the x axis, the y axis, and the z axis to accomplish this task. This individual matrices specified may be multiplied together to give one 4×4 matrix that represents the general rotation.
